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Feasible Region in General Design Space of Lamination Parameters for Laminated Composites

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In the classical lamination theory and the first-order shear deformation theory, the stiffnesses of laminated composites can be expressed as linear functions of 12 lamination parameters. A method is presented for determining the feasible region in general design space of 12 lamination parameters. In some optimization problems, the local optimum can be avoided by using lamination parameters instead of layer angles and thicknesses. Thus, the lamination parameters are useful design variables in the layup optimization for mechanical properties of laminated composites. To perform the layup optimization, the feasible region of the lamination parameters needs to be known. The lamination parameters are functionals of the distribution function of fiber orientation angles through the thickness. In the determination of the feasible region, the laminate configurations are not restricted. In the method, a variational approach is applied to find the boundary of the feasible region in the general design space of 12 lamination parameters. The feasible region for any set of lamination parameters can be also obtained. With use of the method, the feasible regions in four different design spaces are examined as an example. The reliability and the validity of the method are confirmed.

Introduction

IN the layup design of laminated composites, lamination parameters, which characterize laminate configurations (layer angles and layer thicknesses), are useful design variables, and a layup optimization method using lamination parameters is one of the most effective and reliable methods. To use the lamination parameters as design variables in optimization problems, the feasible region in the design space of lamination parameters needs to be known.

In the classical lamination theory, the in-plane, coupling, and out-of-plane stiffness components of laminated composites are expressed as linear functions of four in-plane, four coupling, and four out-of-plane lamination parameters, respectively. These 12 lamination parameters govern the stiffness characteristics of laminated composites.[1] In the specially orthotropic laminated plates and shells that eliminate all coupling terms, the stiffness characteristics are governed by two in-plane and two out-of-plane lamination parameters. Fukunaga and Hirano[2] have used the two out-of-plane lamination parameters as design variables in the buckling design. The fundamental relationship between two in-plane or between two out-of-plane lamination parameters and their feasible region have been derived by Miki[3,4] and by Fukunaga and Chou.[5,6] Fukunaga and Vanderplaats[7] examined the relationship between two in-plane and two out-of-plane lamination parameters for the buckling design of orthotropic laminated cylindrical shells, although the obtained feasible region has been proved to be small in a later study performed by Grenestedt and Gudmundson.[8] Gürdal et al.[9] used lamination parameters to examine discrete optimization problems for laminated composites consisting of layers with equal thickness and restricted layer angles. In the symmetrically laminated plates with extension–shear or bending–twisting coupling, the four in-plane or four out-of-plane lamination parameters can be used as design variables, where the four coupling lamination parameters are vanished due to the midplane symmetry assumption. The explicit feasible region among four in-plane or four out-of-plane lamination parameters has been derived by Fukunaga and Sekine[10] following a few preliminary studies[11,12] that were performed earlier. The lamination parameters have been used successfully as design variables in the optimization problems for vibration,[13] buckling,[14] and topological design.[15]

In the first-order shear deformation theory, the shear stiffness should also be considered. Grenestedt[16] has provided the lamination parameters for Reissner–Mindlin plates, where it was shown that the shear stiffness components can be expressed by two in-plane lamination parameters. Therefore, the feasible region considering the coupling effect between the in-plane and out-of-plane lamination parameters is required even if the laminated plates have symmetric layup. However, the relationship between in-plane and out-of-plane lamination parameters has not been derived so far.

Foldager et al.[17] have proposed a layup optimization approach using the convexity in the feasible region of lamination parameters, where 12 lamination parameters are introduced based on the first-order shear deformation theory. Grédiac[18] has examined another layup design procedure for classical lamination theory in which 12 lamination parameters are used in the objective function for designing laminated plates with required stiffness properties. In both studies[17,18] the design variables are not lamination parameters but layer angles for every layer. Therefore, the investigation into the feasible region of the lamination parameters has not been made.

To obtain the global optimum in the layup design of laminated composites, the lamination parameters should be used as design variables instead of layer angles and layer thicknesses. The effectiveness of the lamination parameters has been shown theoretically in the research work performed by Grenestedt and Gudmundson.[8] In this study, the convexity of the feasible region of lamination parameters is proved when there is no restriction on layer angles and layer thicknesses. Moreover, the feasible region of lamination parameters for the orthotropic laminated shells is obtained analytically using a variational approach.

The present paper shows a method for determining the feasible region in general design space of 12 lamination parameters based on a variational approach. The 12 lamination parameters are considered for a full utilization of lamination parameters in the determination of the feasible region. To clarify the matters that demand special attention in the present method, the feasible region for a simple design space is examined analytically. The feasible regions in three different design spaces for vibration or buckling design of symmetrically laminated composites are also examined numerically as examples

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of practical application. The dependency among the lamination parameters on the feasible region is also discussed by comparing the obtained feasible regions.

Stiffness Characteristics and Lamination Parameters

In the first-order shear deformation theory, the constitutive equations for the laminated composites are expressed in matrix form as

$$\begin{Bmatrix} \boldsymbol{N} \\ \boldsymbol{M} \\ \boldsymbol{Q} \end{Bmatrix} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} & \boldsymbol{0} \\ \boldsymbol{B} & \boldsymbol{D} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \bar{\boldsymbol{A}} \end{bmatrix} \begin{Bmatrix} \boldsymbol{\epsilon}_0 \\ \boldsymbol{\kappa} \\ \boldsymbol{\gamma}_0 \end{Bmatrix} \tag{1}$$

where $\boldsymbol{N}$, $\boldsymbol{M}$, and $\boldsymbol{Q}$ are the stress, moment, and transverse shear stress resultants, respectively; $\boldsymbol{\epsilon}_0$, $\boldsymbol{\kappa}$, and $\boldsymbol{\gamma}_0$ are the strains, the curvatures at the midplane, and the transverse shear strains, respectively; and $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{D}$, and $\bar{\boldsymbol{A}}$ are the in-plane, coupling, out-of-plane, and shear stiffnesses, respectively. In the classical lamination theory, the effect of transverse shear is neglected.

When the stiffness invariants and the lamination parameters are introduced, the stiffness components A_{ij}, B_{ij}, and D_{ij}, $i, j = 1, 2, 6$, and also $\bar{A}_{ij}$, $i, j = 4, 5$, can be expressed as follows[1,8]:

$$\begin{Bmatrix} A_{11} \\ A_{22} \\ A_{12} \\ A_{66} \\ A_{16} \\ A_{26} \end{Bmatrix} = h \begin{bmatrix} 1 & \xi_1^A & \xi_2^A & 0 & 0 \\ 1 & -\xi_1^A & \xi_2^A & 0 & 0 \\ 0 & 0 & -\xi_2^A & 1 & 0 \\ 0 & 0 & -\xi_2^A & 0 & 1 \\ 0 & \xi_3^A/2 & \xi_4^A & 0 & 0 \\ 0 & \xi_3^A/2 & -\xi_4^A & 0 & 0 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{Bmatrix} \tag{2}$$

$$\begin{Bmatrix} B_{11} \\ B_{22} \\ B_{12} \\ B_{66} \\ B_{16} \\ B_{26} \end{Bmatrix} = \frac{h^2}{4} \begin{bmatrix} 0 & \xi_1^B & \xi_2^B & 0 & 0 \\ 0 & -\xi_1^B & \xi_2^B & 0 & 0 \\ 0 & 0 & -\xi_2^B & 0 & 0 \\ 0 & 0 & -\xi_2^B & 0 & 0 \\ 0 & \xi_3^B/2 & \xi_4^B & 0 & 0 \\ 0 & \xi_3^B/2 & -\xi_4^B & 0 & 0 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{Bmatrix} \tag{3}$$

$$\begin{Bmatrix} D_{11} \\ D_{22} \\ D_{12} \\ D_{66} \\ D_{16} \\ D_{26} \end{Bmatrix} = \frac{h^3}{12} \begin{bmatrix} 1 & \xi_1^D & \xi_2^D & 0 & 0 \\ 1 & -\xi_1^D & \xi_2^D & 0 & 0 \\ 0 & 0 & -\xi_2^D & 1 & 0 \\ 0 & 0 & -\xi_2^D & 0 & 1 \\ 0 & \xi_3^D/2 & \xi_4^D & 0 & 0 \\ 0 & \xi_3^D/2 & -\xi_4^D & 0 & 0 \end{bmatrix} \begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{Bmatrix} \tag{4}$$

$$\begin{Bmatrix} \bar{A}_{44} \\ \bar{A}_{55} \\ \bar{A}_{45} \end{Bmatrix} = Kh \begin{bmatrix} 1 & \xi_1^A \\ 1 & -\xi_1^A \\ 0 & -\xi_3^A \end{bmatrix} \begin{Bmatrix} \bar{U}_1 \\ \bar{U}_2 \end{Bmatrix} \tag{5}$$

where h is the thickness of the plate and K is the shear correction factor. The stiffness invariants U_i, $i = 1, 2, 3, 4, 5$, and $\bar{U}_i$, $i = 1, 2$, are material parameters defined as follows:

$$\begin{Bmatrix} U_1 \\ U_2 \\ U_3 \\ U_4 \\ U_5 \end{Bmatrix} = \begin{bmatrix} 3/8 & 3/8 & 1/4 & 1/2 \\ 1/2 & -1/2 & 0 & 0 \\ 1/8 & 1/8 & -1/4 & -1/2 \\ 1/8 & 1/8 & 3/4 & -1/2 \\ 1/8 & 1/8 & -1/4 & 1/2 \end{bmatrix} \begin{Bmatrix} Q'_{11} \\ Q'_{22} \\ Q'_{12} \\ Q'_{66} \end{Bmatrix} \tag{6}$$

$$\begin{Bmatrix} \bar{U}_1 \\ \bar{U}_2 \end{Bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} \begin{Bmatrix} Q'_{44} \\ Q'_{55} \end{Bmatrix} \tag{7}$$

where

$$Q'_{11} = E_{11}/(1 - \nu_{12}\nu_{21}), \qquad Q'_{22} = E_{22}/(1 - \nu_{12}\nu_{21})$$
$$Q'_{12} = \nu_{21}Q'_{11} = \nu_{12}Q'_{22}, \qquad Q'_{44} = G_{23}$$
$$Q'_{55} = G_{31}, \qquad Q'_{66} = G_{12} \tag{8}$$

In Eq. (8), E_{11}, E_{22}, G_{12}, G_{23}, G_{31}, and ν_{12} are engineering constants of a unidirectional laminate. The lamination parameters in Eqs. (2–5) are defined as follows:

$$\xi^A_{[1,2,3,4]} = \frac{1}{2}\int_{-1}^{1} [\cos 2\theta(\bar{z}), \cos 4\theta(\bar{z}), \sin 2\theta(\bar{z}), \sin 4\theta(\bar{z})]\,\mathrm{d}\bar{z} \tag{9}$$

$$\xi^B_{[1,2,3,4]} = \int_{-1}^{1} [\cos 2\theta(\bar{z}), \cos 4\theta(\bar{z}), \sin 2\theta(\bar{z}), \sin 4\theta(\bar{z})]\bar{z}\,\mathrm{d}\bar{z} \tag{10}$$

$$\xi^D_{[1,2,3,4]} = \frac{3}{2}\int_{-1}^{1} [\cos 2\theta(\bar{z}), \cos 4\theta(\bar{z}), \sin 2\theta(\bar{z}), \sin 4\theta(\bar{z})]\bar{z}^2\,\mathrm{d}\bar{z} \tag{11}$$

where $\xi^A_{1,2,3,4}$, $\xi^B_{1,2,3,4}$, and $\xi^D_{1,2,3,4}$ are the in-plane, coupling, and out-of-plane lamination parameters, respectively, and $\theta(\bar{z})$ is the distribution function of the fiber orientation angles through the normalized thickness $\bar{z}(= 2z/h)$. Note that these 12 lamination parameters are not independent because they are functionals of the layup function $\theta(\bar{z})$. In the symmetric laminates, the relationships among $\xi^A_{1,2,3,4}$ or among $\xi^D_{1,2,3,4}$ were obtained analytically in the literature.[10] The relationships among $\xi^A_{1,2,3,4}$ are expressed as follows:

$$2\left(\xi_1^A\right)^2 - 1 \le \xi_2^A \le 1 - 2\left(\xi_3^A\right)^2$$
$$2\left(1 + \xi_2^A\right)\left(\xi_3^A\right)^2 - 4\xi_1^A\xi_3^A\xi_4^A + \left(\xi_4^A\right)^2$$
$$\le \left[\xi_2^A - 2\left(\xi_1^A\right)^2 + 1\right]\left(1 - \xi_2^A\right) \tag{12}$$

The relationships among $\xi^D_{1,2,3,4}$ for the symmetric laminates are obtained by replacing $\xi^A_{1,2,3,4}$ with $\xi^D_{1,2,3,4}$ in Eq. (12). These relationships show the feasible region in the design space of the lamination parameters.

In the first-order shear deformation theory, as well as in the classical lamination theory, all stiffness components are linear functions of the lamination parameters. Thus, the nonlinearity between the objective functions (stiffnesses, vibration frequencies, or buckling loads) and the design variables can be reduced remarkably, and the optimization problems are easy to solve by an introduction of lamination parameters.

In the optimization method using lamination parameters as design variables, the feasible region of the lamination parameters needs to be known because the feasible region should be considered as a constraint. The feasible region of lamination parameters has been obtained analytically only for some specific sets of lamination parameters.[7,8] The feasible region for other sets of lamination parameters has remained unknown. This lack of knowledge of the feasible region is an obstacle in the use of the lamination parameters as design variables.

Feasible Region Determination

Various sets of lamination parameters are necessary for different optimization problems. In this section, we deal with a variational method[8] extended to determine the feasible region for any desired set of lamination parameters.

In the general design space of all 12 lamination parameters, the boundary of the feasible region is obtained by determining the layup function $\theta(\bar{z})$ that maximizes the following functional:

$$F[\theta(\bar{z})] = k_1^A\xi_1^A + k_2^A\xi_2^A + k_3^A\xi_3^A + k_4^A\xi_4^A + k_1^B\xi_1^B + k_2^B\xi_2^B + k_3^B\xi_3^B$$
$$+ k_4^B\xi_4^B + k_1^D\xi_1^D + k_2^D\xi_2^D + k_3^D\xi_3^D + k_4^D\xi_4^D = \int_{-1}^{1} G[\bar{z}, \theta(\bar{z})]\,\mathrm{d}\bar{z} \tag{13}$$

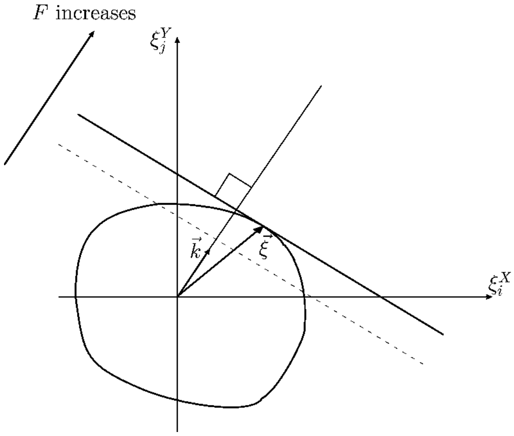


Fig. 1 Feasible region of lamination parameters.

where

$$G[\bar{z}, \theta(\bar{z})] = \sum_{i=1}^{4} g_i(\bar{z}) f_i[\theta(\bar{z})] \tag{14}$$

$$g_i(\bar{z}) = \tfrac{1}{2}k_i^A + k_i^B \bar{z} + \tfrac{3}{2}k_i^D \bar{z}^2 \tag{15}$$

$$f_{[1,2,3,4]}[\theta(\bar{z})] = [\cos 2\theta(\bar{z}), \cos 4\theta(\bar{z}), \sin 2\theta(\bar{z}), \sin 4\theta(\bar{z})] \tag{16}$$

$$\sum_{i=1}^{4} \left[\left(k_i^A\right)^2 + \left(k_i^B\right)^2 + \left(k_i^D\right)^2 \right] = 1 \tag{17}$$

In a geometrical interpretation, F is constant on a hyperplane whose unit normal is $\boldsymbol{k} \equiv (k_1^A, \ldots, k_4^A, k_1^B, \ldots, k_4^B, k_1^D, \ldots, k_4^D)$. The hyperplane is translated in the normal direction when F increases. The limit of the feasible region of the lamination parameters is reached at maximum F for given $\boldsymbol{k}$. In this case, the hyperplane comes into contact with the boundary of the feasible region, as shown in Fig. 1. Because of the convexity of the feasible region of lamination parameters,[8] the boundary of the feasible region is determined by obtaining the hyperplanes for all directions $\boldsymbol{k}$.

The problem is to obtain the layup function $\theta(\bar{z})$ that maximize F for given $\boldsymbol{k}$. Because the layup function $\theta(\bar{z})$ can be discontinuous, the Euler equation has no meaning for this problem. An alternative formulation for determining $\theta(\bar{z})$ to maximize F is given as follows:

$$F[\theta(\bar{z})] - F[\theta(\bar{z}) + \phi(\bar{z})] \geq 0 \tag{18}$$

where $\phi(\bar{z})$ is a piecewise continuous test function chosen as a narrow peak at $\bar{z}_0 \in [-1, 1)$:

$$\phi(\bar{z}) = \Delta[H(\bar{z} - \bar{z}_0) - H(\bar{z} - \bar{z}_0 - \delta)] \tag{19}$$

where H is the Heaviside step function and δ and Δ are the width and the height of the peak. The layup function $\theta(\bar{z})$ and the test function $\phi(\bar{z})$ are shown in Fig. 2, where δ is infinitesimal and positive, whereas Δ is an arbitrary angle.

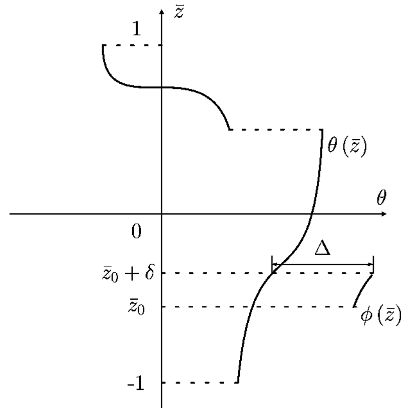


Fig. 2 Layup function $\theta(\bar{z})$ and test function $\phi(\bar{z})$.

When the test function is introduced, the requirement for determining local $\theta(\bar{z}_0)$ is derived as follows:

$$\begin{aligned} &F[\theta(\bar{z})] - F[\theta(\bar{z}) + \phi(\bar{z})] \\ &= \int_{-1}^{1} \{G[\bar{z}, \theta(\bar{z})] - G[\bar{z}, \theta(\bar{z}) + \phi(\bar{z})]\}\, \mathrm{d}\bar{z} \\ &= \int_{\bar{z}_0}^{\bar{z}_0+\delta} \{G[\bar{z}, \theta(\bar{z})] - G[\bar{z}, \theta(\bar{z}) + \Delta]\}\, \mathrm{d}\bar{z} \\ &\to \delta\, \{G[\bar{z}_0, \theta(\bar{z}_0)] - G[\bar{z}_0, \theta(\bar{z}_0) + \Delta]\} \\ &\equiv \frac{1}{2}\delta R \geq 0 \end{aligned} \tag{20}$$

The value of R should be nonnegative for arbitrary Δ to maximize G at local $\bar{z}_0$. When trigonometric functions are used, R can be expressed as follows:

$$R = 2g_1(c - c_\Delta) + 4g_2\left(c^2 - c_\Delta^2\right) + 2g_3(s - s_\Delta) + 4g_4(cs - c_\Delta s_\Delta) \tag{21}$$

where

$$g_i = g_i(\bar{z}_0) \tag{22}$$

$$c = \cos[2\theta(\bar{z}_0)], \qquad c_\Delta = \cos[2\theta(\bar{z}_0) + 2\Delta]$$
$$s = \sin[2\theta(\bar{z}_0)], \qquad s_\Delta = \sin[2\theta(\bar{z}_0) + 2\Delta] \tag{23}$$

Because the value of Δ is arbitrary, (c_Δ, s_Δ) takes an arbitrary point on a unit circle. In the following expression, the value of R should be nonnegative for arbitrary $c_\Delta \in [-1, 1]$:

$$R = \Psi(c) - \Psi(c_\Delta) \geq 0 \tag{24}$$

where

$$\Psi(c) = \begin{cases} 2g_1c + 4g_2c^2 + 2g_3\sqrt{1-c^2} + 4g_4c\sqrt{1-c^2} \equiv \Psi_1(c), & s \geq 0 \\ 2g_1c + 4g_2c^2 - 2g_3\sqrt{1-c^2} - 4g_4c\sqrt{1-c^2} \equiv \Psi_2(c), & s \leq 0 \end{cases} \tag{25}$$

$$\Psi(c_\Delta) = \begin{cases} 2g_1c_\Delta + 4g_2c_\Delta^2 + 2g_3\sqrt{1-c_\Delta^2} + 4g_4c_\Delta\sqrt{1-c_\Delta^2} \equiv \Psi_1(c_\Delta), & s_\Delta \geq 0 \\ 2g_1c_\Delta + 4g_2c_\Delta^2 - 2g_3\sqrt{1-c_\Delta^2} - 4g_4c_\Delta\sqrt{1-c_\Delta^2} \equiv \Psi_2(c_\Delta), & s_\Delta \leq 0 \end{cases} \tag{26}$$

In the case of $s = 0$, the value of $\Psi_1(c)$ is equal to the value of $\Psi_2(c)$, and $\Psi_1(c_\Delta) = \Psi_2(c_\Delta)$ for $s_\Delta = 0$. When the variety of $\Psi(c_\Delta)$ due to the arbitrariness of c_Δ is considered, the following R^* should be nonnegative:

$$R^* = \Psi(c) - \Psi^{\max} \ge 0 \tag{27}$$

where $\Psi^{\max}$ shows the maximum value of $\Psi(c_\Delta)$. When R^* is nonnegative, R is also nonnegative for arbitrary c_Δ. R^* can be classified as follows:

$$R^* = \begin{cases} \Psi(c) - \Psi_1^{\max} \ge 0, & \Psi_1^{\max} > \Psi_2^{\max} \\ \Psi(c) - \Psi_2^{\max} \ge 0, & \Psi_1^{\max} < \Psi_2^{\max} \\ \Psi(c) - \Psi_1^{\max} = \Psi(c) - \Psi_2^{\max} \ge 0, & \Psi_1^{\max} = \Psi_2^{\max} \end{cases} \tag{28}$$

where $\Psi_1^{\max}$ and $\Psi_2^{\max}$ are the maximum value of $\Psi_1(c_\Delta)$ and $\Psi_2(c_\Delta)$, respectively. When $\Psi_1^{\max}$ is greater than $\Psi_2^{\max}$, then $\Psi^{\max}$ is given by $\Psi_1^{\max}$. In this case, the only c^* for maximizing $\Psi(c) = \Psi_1(c)$ fulfills the requirement of $R^* \ge 0$. On the other hand, when $\Psi_2^{\max}$ is greater than $\Psi_1^{\max}$, only the value of c^* for maximizing $\Psi(c) = \Psi_2(c)$ fulfills $R^* \ge 0$. Note, when $\Psi_1(c)$ or $\Psi_2(c)$ has its maximum at more than one point, we have a nonunique solution of c^*. In the case of $\Psi_1^{\max} = \Psi_2^{\max}$, we also have nonunique solution of c^* because c^* for maximizing both $\Psi_1(c)$ and $\Psi_2(c)$ fulfills $R^* \ge 0$. Therefore, the value of c^* that gives maximum value of $\Psi(c)$ fulfills the requirement.

The value of c^* can be found by solving the following extremum condition:

$$\frac{\partial \Psi_1(c)}{\partial c} = 0, \qquad \frac{\partial \Psi_2(c)}{\partial c} = 0 \tag{29}$$

Equation (29) leads to the following quartic equation with respect to c:

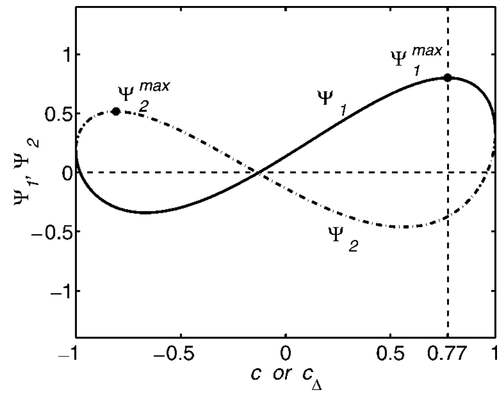
$$16\left(g_2^2 + g_4^2\right)c^4 + 8(g_1 g_2 + g_3 g_4)c^3 + \left[\left(g_1^2 + g_3^2\right) - 16\left(g_2^2 + g_4^2\right)\right]c^2 - 4(2g_1 g_2 + g_3 g_4)c - \left(g_1^2 - 4g_4^2\right) = 0 \tag{30}$$

From the real roots $c_r \in [-1, 1]$ of Eq. (30), c^* is selected to obtain the maximum value of $\Psi(c)$. The local layer angle $\theta(\bar{z}_0)$ is determined from the value of c^* considering the sign of s^*. The layup function $\theta(\bar{z})$ is obtained by determining $\theta(\bar{z}_0)$ for every infinitesimal interval δ on $\bar{z}_0 \in [-1, 1]$).

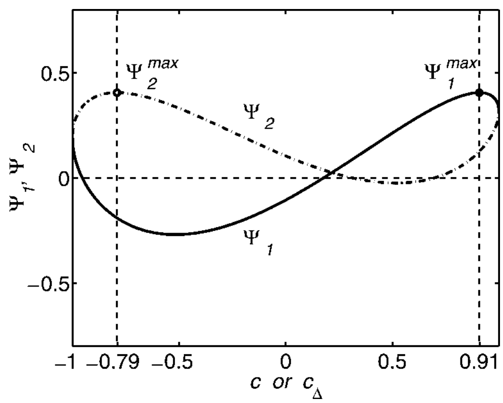
Typical examples of the characteristics of $\Psi_{1,2}$ are shown in Fig. 3, where $\boldsymbol{k} = (0.16, 0.20, -0.32, 0.36, -0.16, 0.24, 0.32, 0.36, 0.20, -0.24, 0.36, 0.40)$ is given. The values of $\Psi_{1,2}$ at $\bar{z}_0 = 0.5$ and at $\bar{z}_0 = 0.14$ are shown in Figs. 3a and 3b, respectively. In Fig. 3a, $\Psi_1^{\max} > \Psi_2^{\max}$ and $R^* = \Psi(c) - \Psi_1^{\max}$ should be nonnegative. The value of R^* is nonnegative at $c^* = 0.77$, which maximizes $\Psi(c) = \Psi_1(c)$, where $R^* = 0$. From the value of c^*, a local layer angle is determined as $\theta(0.5) = 19.8$ deg because s^* is nonnegative for $\Psi_1(c)$. In Fig. 3b it can be seen that $\Psi_1^{\max} = \Psi_2^{\max}$. Hence, $R^* = \Psi(c) - \Psi_1^{\max}$ and $R^* = \Psi(c) - \Psi_2^{\max}$ should be nonnegative. The value of $c^* = 0.91$ that maximizes $\Psi(c) = \Psi_1(c)$ gives a nonnegative value of $R^* = \Psi(c) - \Psi_1^{\max}$, and $c^* = -0.79$ that maximizes $\Psi(c) = \Psi_2(c)$ gives a nonnegative value of $R^* = \Psi(c) - \Psi_2^{\max}$. In this case, there exists a double solution of c^*, which yields $\theta(0.14) = -12.25$ and $\theta(0.14) = 71.1$ deg, respectively. This means that the layup function $\theta(\bar{z})$ has a discontinuity at the position $\bar{z}_0 = 0.14$. The layup function through the thickness is also shown in Fig. 4. In Fig. 4, the discontinuity of $\theta(\bar{z})$ occurs not only at $\bar{z}_0 = 0.14$ but also at $\bar{z}_0 = -0.68$.

Depending on $\bar{z} = \bar{z}_0$, the two different c^* that maximize $\Psi(c)$ can be found and yield a double solution of $\theta(\bar{z}_0)$. This nonuniqueness of $\theta(\bar{z}_0)$ occurs only at a special position of $\bar{z}_0$ in which the layup function has a discontinuity. In this case, the layup function and lamination parameters are determined uniquely because the nonuniqueness occurs locally on infinitesimal interval δ.

On the other hand, there exist some cases where the nonuniqueness of $\theta(\bar{z}_0)$ provides a nonunique solution of the layup function $\theta(\bar{z})$. In the case of $k_{1,4}^A = k_{1,4}^B = k_{1,4}^D = 0$, for example, $\Psi_1(c)$ or $\Psi_2(c)$ can have two different maximum points depending on $\bar{z}_0$. The double solution of c^* for maximizing $\Psi_1(c)$ or $\Psi_2(c)$ provides a double solution of $\theta(\bar{z}_0)$. In this case, the nonuniqueness of $\theta(\bar{z}_0)$ occurs at a partial region in the thickness of the laminate. In the



a) $\bar{z}_0 = 0.5$



b) $\bar{z}_0 = 0.14$

Fig. 3 Characteristics of Ψ_1 and Ψ_2.

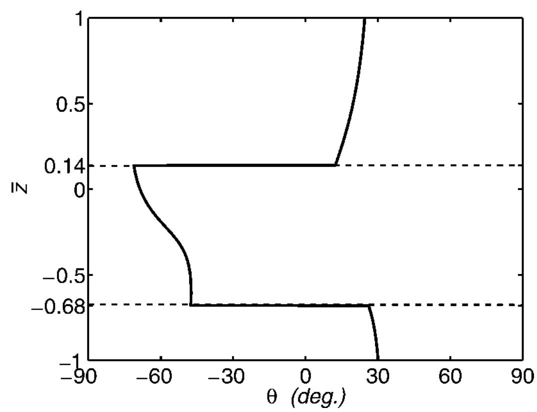


Fig. 4 Layup function through the thickness $\theta(\bar{z})$.

case of $k_{1,3}^A = k_{1,3}^B = k_{1,3}^D = 0$, the double solution of c^* is obtained for every infinitesimal interval δ on $\bar{z}_0 \in [-1, 1)$ because the relation $\Psi_1^{\max} = \Psi_2^{\max}$ holds at any position of $\bar{z}_0$. From the double solution of c^*, two different angles $\theta(\bar{z}_0)$ are determined when considering the sign of s^*. The nonuniqueness of $\theta(\bar{z}_0)$ through the thickness also occurs in the case of $k_{3,4}^A = k_{3,4}^B = k_{3,4}^D = 0$. In this case, the unique c^* gives $\pm\theta(\bar{z}_0)$ because the dependency on the sign of s has disappeared from the requirement in Eq. (28). When $\pm\theta(\bar{z}_0)$ is chosen for every infinitesimal interval δ on $\bar{z}_0 \in [-1, 1)$, an orthotropic laminate is obtained. In the case of $m_1 k_i^A = m_2 k_i^B = m_3 k_i^D$, $i = 1, 2, 3, 4$, where m_1, m_2, and m_3 are constants, the nonuniqueness of $\theta(\bar{z}_0)$ can also happen because the dependency on $\bar{z}_0$ is lost in the requirement in Eq. (28). In those cases, the layup function $\theta(\bar{z})$ cannot be determinate uniquely due to the nonuniqueness of $\theta(\bar{z}_0)$. When the nonunique solutions of $\theta(\bar{z})$ provide nonunique sets of lamination parameters, the boundary of the feasible region is geometrically flat in the given direction $\boldsymbol{k}$. However, the hyperplane F remains unique in spite of the nonuniqueness of layup function $\theta(\bar{z})$ and the implicit nonuniqueness of the lamination parameters.

From the general formulation in Eq. (13), the feasible region for any set of lamination parameters can be obtained. When a desired set of lamination parameters is specified, some of the undesired lamination parameters can be neglected due to their arbitrariness, and others are vanished by imposing some conditions on $\theta(\bar{z})$ such as symmetry or orthotropy.

Feasible Region in Design Space of Lamination Parameters

In the preceding section, a method for determining the boundary of feasible region in the design space of all 12 lamination parameters is shown. In this section, we will apply the present method to obtain the feasible region in several design spaces of lamination parameters. First, the feasible region for simple set of only two lamination parameters will be obtained analytically to clarify the relation between the nonunique solution of $\theta(\bar{z})$ and the feasible region. Next, the feasible region for three different sets of lamination parameters will be obtained numerically when considering the vibration or buckling design of symmetrically laminated composites as examples.

Feasible Region for $\xi^D_{3,4}$

The feasible region in two-dimensional design space $\xi^D_{3,4}$ is considered, where other lamination parameters are allowed to take arbitrary values. The functional in Eq. (13) becomes

$$F = k_3^D \xi_3^D + k_4^D \xi_4^D \tag{31}$$

and the local requirement at $\bar{z}_0$ is obtained from Eq. (28). The following $\Psi(c)$ can be obtained from Eq. (25) (the upper notation of $k^D_{3,4}$ is neglected from now on):

$$\Psi(c) = \begin{cases} 3\bar{z}_0^2\left[k_3\sqrt{1-c^2} + 2k_4 c\sqrt{1-c^2}\right] \equiv \Psi_1(c), & s \geq 0 \\ -\Psi_1(c) \equiv \Psi_2(c), & s \leq 0 \end{cases} \tag{32}$$

The value of c^* that maximizes $\Psi(c)$ should be determined. In Eq. (32), c^* that maximizes $\Psi(c)$ is determined independent of $\bar{z}_0$. This means that the layup function $\theta(\bar{z})$ is constant through the thickness for a single solution c^* in the case of $\Psi_1^{\max} \neq \Psi_2^{\max}$ and that the laminate is unidirectional. Otherwise, when $\Psi_1^{\max} = \Psi_2^{\max}$, $\theta(\bar{z})$ will be any combination of two angles corresponding to the double solution of c^* that gives $\Psi_1^{\max}$ and $\Psi_2^{\max}$.

The solution c^* is selected from the real roots of Eq. (30). For present case, Eq. (30) becomes

$$4k_4c^2 + k_3c - 2k_4 = 0 \tag{33}$$

Because $k_4 = \pm\sqrt{(1-k_3^2)}$ from the condition in Eq. (17), the real roots c_r of Eq. (33) are given as follows:

$$c_r = \begin{cases} \dfrac{-k_3 \pm \sqrt{32 - 31k_3^2}}{\pm 8\sqrt{1-k_3^2}}, & k_4 \geq 0 \\ \dfrac{k_3 \mp \sqrt{32 - 31k_3^2}}{\pm 8\sqrt{1-k_3^2}}, & k_4 < 0 \end{cases} \tag{34}$$

Figure 5 shows the characteristics of Ψ_1 and Ψ_2 for every k_3 in the case of $k_4 \geq 0$. In cross sections, the solid and dashed lines show Ψ_1 and Ψ_2, respectively, and the dots show the values of c^* and maximum values of $\Psi(c)$ that fulfill the requirement. The maximum value of $\Psi(c)$ is given by $\Psi_2^{\max}$ for $-1 \leq k_3 < 0$ and given by $\Psi_1^{\max}$ for $0 < k_3 \leq 1$. In the case of $k_3 = 0$, $\Psi_1^{\max} = \Psi_2^{\max}$ is given, and layup angle is not unique.

Because $\boldsymbol{k} = (k_3, k_4)$ act on a unit circle, the relation between (k_3, k_4) and layup angle θ can be obtained from Table 1. For example, in the case of $\boldsymbol{k} = (0, 1)$, $\Psi_1^{\max} = \Psi_2^{\max}$ are given for $c^* = \pm\sqrt{(2)}/2$, and $\theta = -67.5$ or 22.5 deg is obtained. On the other hand, in the case of $\boldsymbol{k} = (0, -1)$, $\Psi_1^{\max} = \Psi_2^{\max}$ at $c^* = \pm\sqrt{(2)}/2$ is also observed, but $\theta = -22.5$ or 67.5 deg is obtained when considering the sign of s^*. In these cases, the uniqueness of θ is lost, and the boundary of feasible region is geometrically flat because the value of ξ_3^D can take different values as a combination of two given angles. The obtained feasible region for $\xi^D_{3,4}$ is shown in Fig. 6.

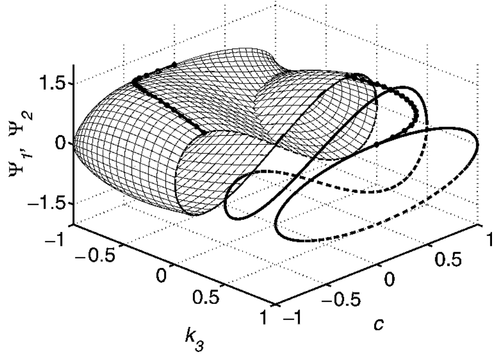


Fig. 5 Ψ_1 and Ψ_2 for $k_4 \geq 0$.

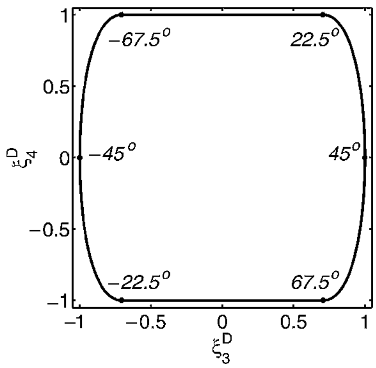


Fig. 6 Feasible region for $\xi^D_{3,4}$ ($\xi^A_{1,2,3,4}$, $\xi^B_{1,2,3,4}$, $\xi^D_{1,2}$: arbitrary).

This simple analysis also can be done for other sets of two lamination parameters. The only necessary condition is the two lamination parameters should be selected from the same category, that is, in-plane, coupling, or out-of-plane. Note that for sets of two coupling lamination parameters we should pay attention to the sign of $\bar{z}_0$ in the analytic investigation. For three or four lamination parameters selected from the same category, it is much more difficult to obtain the analytic solution because Eq. (30) is quartic. Moreover, when the feasible region for a set of lamination parameters selected from a different category is needed, a numerical procedure is necessary to process the local requirement through the thickness for small intervals δ.

Next, numerical examples will be given for symmetric laminated plates. Hence, the four terms in Eq. (13) concerned with the coupling lamination parameters vanish because $\xi^B_{1,2,3,4} = 0$ due to the symmetric condition at the midplane.

Feasible Region for $\xi^D_{1,2,3,4}$

Based on the classical lamination theory, the vibration frequencies or buckling loads for symmetric laminated plates depend on the out-of-plane stiffnesses D_{ij}, $i, j = 1, 2, 6$. Thus, the four out-of-plane lamination parameters $\xi^D_{1,2,3,4}$ that govern the stiffnesses D_{ij} can be used as design variables, whereas the in-plane lamination parameters $\xi^A_{1,2,3,4}$ are allowed to take arbitrary values. Therefore, the feasible region in the four-dimensional design space of $\xi^D_{1,2,3,4}$ is needed. The feasible region for $\xi^D_{1,2,3,4}$ is obtained by maximizing the following functional F for all $\boldsymbol{k} = (k_1^D, k_2^D, k_3^D, k_4^D)$:

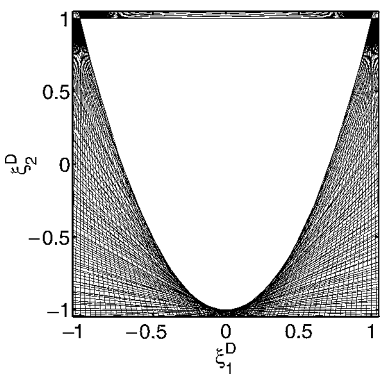
$$F = k_1^D\xi_1^D + k_2^D\xi_2^D + k_3^D\xi_3^D + k_4^D\xi_4^D \tag{35}$$

Figure 7 shows the feasible region obtained by the present method, where 36×10^4 sets of $\boldsymbol{k}$ generated by a random number generator are used. The obtained feasible region in the four-dimensional space $\xi^D_{1,2,3,4}$ is represented on the two-dimensional lamination parameters plane $\xi_1^D - \xi_2^D$ and $\xi_3^D - \xi_4^D$. Figures 7a and 7b show the feasible region of (ξ_1^D, ξ_2^D) for fixed value of $\xi^D_{3,4} = 0$ and the feasible region of (ξ_3^D, ξ_4^D) for the fixed value of $\xi^D_{1,2} = 0$, respectively.

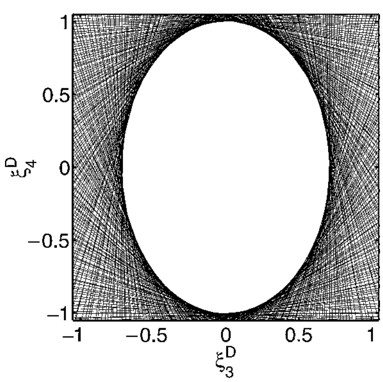
The feasible region among the four out-of-plane lamination parameters can be obtained from the relationships that replacing $\xi^A_{1,2,3,4}$ with $\xi^D_{1,2,3,4}$ in Eq. (12). The exact feasible regions corresponding

Table 1 Layup angle θ depending on directions k_3 and k_4

Direction	$k_3 = -1$	$-1 < k_3 < 0$	$k_3 = 0$	$0 < k_3 < 1$	$k_3 = 1$
$k_4 = 1$			$\Psi_1^{max} = \Psi_2^{max}$ $\theta = -67.5$ or 22.5 deg		
$0 < k_4 < 1$		$\Psi_1^{max} < \Psi_2^{max}$ $\theta \in (-67.5, -45)$ deg		$\Psi_1^{max} > \Psi_2^{max}$ $\theta \in (22.5, 45)$ deg	
$k_4 = 0$	$\Psi_1^{max} < \Psi_2^{max}$ $\theta = -45$ deg				$\Psi_1^{max} > \Psi_2^{max}$ $\theta = 45$ deg
$-1 < k_4 < 0$		$\Psi_1^{max} < \Psi_2^{max}$ $\theta \in (-45, -22.5)$ deg		$\Psi_1^{max} > \Psi_2^{max}$ $\theta \in (45, 67.5)$ deg	
$k_4 = -1$			$\Psi_1^{max} = \Psi_2^{max}$ $\theta = -22.5$ or 67.5 deg		

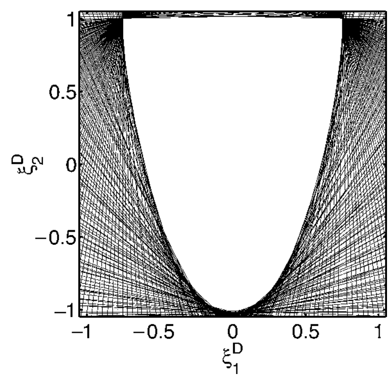


a) (ξ_1^D, ξ_2^D) plane for $\xi_{3,4}^D = 0$

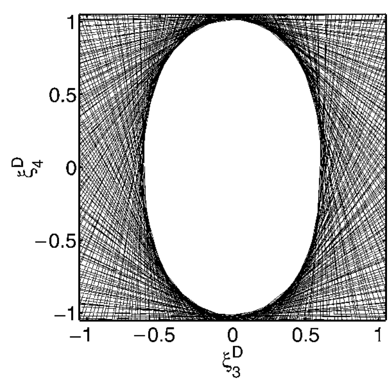


b) (ξ_3^D, ξ_4^D) plane for $\xi_{1,2}^D = 0$

Fig. 7 Feasible region for $\xi_{1,2,3,4}^D$ ($\xi_{1,2,3,4}^A$: arbitrary, $\xi_{1,2,3,4}^B = 0$).



a) (ξ_1^D, ξ_2^D) plane for $\xi_{1,3}^A = \xi_{3,4}^D = 0$



b) (ξ_3^D, ξ_4^D) plane for $\xi_{1,3}^A = \xi_{1,2}^D = 0$

Fig. 8 Feasible region for $\xi_{1,3}^A, \xi_{1,2,3,4}^D$ ($\xi_{2,4}^A$: arbitrary, $\xi_{1,2,3,4}^B = 0$).

to the Figs. 7a and 7b are given by $2(\xi_1^D)^2 - 1 \leq \xi_1^D \leq 1$ and $2(\xi_3^D)^2 + (\xi_4^D)^2 \leq 1$, respectively. The feasible regions in Figs. 7a and 7b coincide with those formulas. This result means that the present method confirms other results obtained in the literature[10] and provides the feasible region of the lamination parameters properly.

Feasible Region for $\xi_{1,3}^A, \xi_{1,2,3,4}^D$

Based on the first-order shear deformation theory, the vibration frequencies or buckling loads for symmetric laminated plates depend on both the stiffnesses D_{ij}, $i, j = 1, 2, 6$, and $\bar{A}_{ij}$, $i, j = 4, 5$. In this case, the six lamination parameters $\xi_{1,3}^A, \xi_{1,2,3,4}^D$ can be used as design variables, whereas $\xi_{2,4}^A$ are allowed to take arbitrary values. Thus, the feasible region in six-dimensional design space of $\xi_{1,3}^A, \xi_{1,2,3,4}^D$ is needed. The feasible region for $\xi_{1,3}^A, \xi_{1,2,3,4}^D$ is obtained by maximizing the following functional F for all $\boldsymbol{k} = (k_1^A, k_3^A, k_1^D, k_2^D, k_3^D, k_4^D)$:

$$F = k_1^A \xi_1^A + k_3^A \xi_3^A + k_1^D \xi_1^D + k_2^D \xi_2^D + k_3^D \xi_3^D + k_4^D \xi_4^D \tag{36}$$

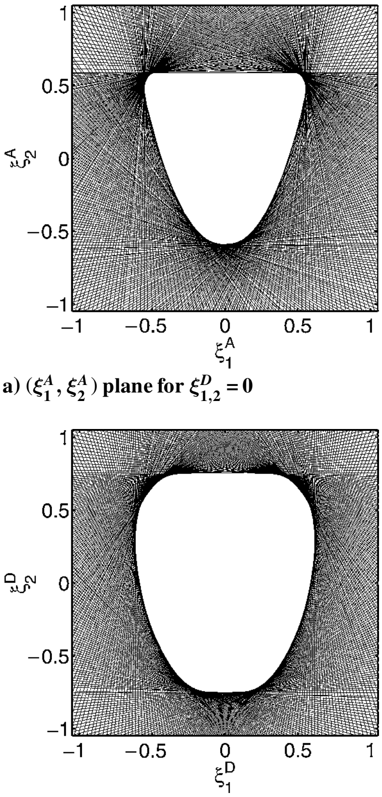
Figure 8 shows the feasible region obtained by the present method, where 36×10^6 sets of $\boldsymbol{k}$ random generated are used. In Fig. 8, the obtained feasible region in the six-dimensional design space of $\xi_{1,3}^A, \xi_{1,2,3,4}^D$ is represented on the two-dimensional lamination parameters plane $\xi_1^D - \xi_2^D$ and $\xi_3^D - \xi_4^D$. Figures 8a and 8b show the feasible region of (ξ_1^D, ξ_2^D) for fixed values of $\xi_{1,3}^A = \xi_{3,4}^D = 0$ and the feasible region of (ξ_3^D, ξ_4^D) for the fixed values of $\xi_{1,3}^A = \xi_{1,2}^D = 0$, respectively.

When Fig. 8 is compared with Fig. 7, the feasible region of (ξ_1^D, ξ_2^D) shown in Fig. 8a is smaller than that in Fig. 7a due to an additional condition of $\xi_{1,3}^A = 0$. This additional condition also cause the reduction in the feasible region of (ξ_3^D, ξ_4^D) as shown in Fig. 8b. This shows that there exists a strong dependency between $\xi_{1,2,3,4}^D$ and $\xi_{1,3}^A$. Note that the additional condition $\xi_{1,3}^A = 0$ has special influence on the directions ξ_1^D and ξ_3^D.

Applying the present method to the determination of the feasible region for $\xi_{1,3}^A, \xi_{1,2,3,4}^D$, the design of symmetric laminated plates based on the first-order shear deformation theory can be undertaken by using lamination parameters as design variables.

Feasible Region for $\xi_{1,2}^A, \xi_{1,2}^D$

In the orthotropic laminated cylindrical shells, the vibration frequencies or buckling loads for orthotropic laminated plates depend



a) (ξ_1^A, ξ_2^A) plane for $\xi_{1,2}^D = 0$

b) (ξ_1^D, ξ_2^D) plane for $\xi_{1,2}^A = 0$

Fig. 9 Feasible region for $\xi_{1,2}^A, \xi_{1,2}^D$ ($\xi_{3,4}^A = 0, \xi_{1,2,3,4}^B = 0, \xi_{3,4}^D = 0$).

on both the stiffnesses $D_{ij}, i, j = 1, 2, 6$, and $A_{ij}, i, j = 1, 2, 6$, where $A_{i6} = D_{i6} = 0, i = 1, 2$. Thus, the four lamination parameters $\xi_{1,2}^A, \xi_{1,2}^D$ can be used as design variables under the orthotropic condition of $\xi_{3,4}^A = \xi_{3,4}^D = 0$. In this case, $\xi_{3,4}^A, \xi_{3,4}^D$ are not allowed to take arbitrary values. However, this condition can be imposed by the proper choice of the layup function $\theta(\bar{z})$, that is, for every $\bar{z}_0 \in [-1, 1)$ a $\pm\theta(\bar{z}_0)$ angle ply is considered. Therefore, only $\xi_{1,2}^A, \xi_{1,2}^D$ are needed as design variables. The feasible region for $\xi_{1,2}^A, \xi_{1,2}^D$ is obtained by maximizing the following functional F for $\boldsymbol{k} = (k_1^A, k_2^A, k_1^D, k_2^D)$:

$$F = k_1^A \xi_1^A + k_2^A \xi_2^A + k_1^D \xi_1^D + k_2^D \xi_2^D \tag{37}$$

Figure 9 shows the feasible region obtained by the present method, where 36×10^4 sets of $\boldsymbol{k}$ were generated randomly. In Fig. 9, the obtained feasible region in the four-dimensional design space $\xi_{1,2}^A, \xi_{1,2}^D$ represented on two-dimensional lamination parameters plane $\xi_1^A - \xi_2^A$ and $\xi_1^D - \xi_2^D$. Figures 9a and 9b show the feasible region of (ξ_1^A, ξ_2^A) for $\xi_{3,4}^A = \xi_{1,2,3,4}^D = 0$ and the feasible region of (ξ_1^D, ξ_2^D) for $\xi_{1,2,3,4}^A = \xi_{3,4}^D = 0$, respectively. The feasible region shown in Fig. 9b is the same as the region obtained in the literature.[8]

The influence of the new imposed condition $\xi_{2,4}^A = 0$ can be observed in Fig. 9b compared to Fig. 8a. In Fig. 8a, $\xi_{2,4}^A$ were allowed to take arbitrary values. In Fig. 9b, the size of the feasible region decreases along the direction ξ_2^D. In Fig. 9, the relation between (ξ_1^A, ξ_2^A) and (ξ_1^D, ξ_2^D) can be observed. When Fig. 9a is compared with Fig. 9b, the feasible region of (ξ_1^A, ξ_2^A) in Fig. 9a is smaller than that of (ξ_1^D, ξ_2^D) in Fig. 9b. This shows that the effect of (ξ_1^D, ξ_2^D) on (ξ_1^A, ξ_2^A) and the effect of (ξ_1^A, ξ_2^A) on (ξ_1^D, ξ_2^D) is not the same.

When the feasible region for $\xi_{1,2}^A, \xi_{1,2}^D$ is obtained by using the present method, the design of the orthotropic laminated cylindrical shells can be done in its proper design space.

Conclusions

Based on a variational approach, a method for determining the feasible region in general design space of 12 lamination parameters is presented. The present method can provide the feasible region for any set of lamination parameters. The feasible region for two lamination parameters in the same category can be obtained analytically, and a comprehensive explanation about the flat region on the boundary is given. In a numerical example, the feasible region among four out-of-plane lamination parameters is obtained. The result is confirmed by the analytic relationships between lamination parameters. The feasible region that has remained unknown is also obtained, and the effect of dependency among the lamination parameters on the feasible region is clarified. The present method does not give any explicit relationship between the lamination parameters, but the boundary of the feasible region in any design space of lamination parameters can be built numerically.

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